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Absorption of Elastic Waves - An Operator for a Constant Q Mechanism

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SUMMARY

Theoretical analysis is given of absorption of elastic waves in the frequency range of seismological interest. The impulse response of a system satisfying the constant Q hypothesis is calculated and the results discussed.

1. INTRODUCTION

Over the past few years much has been written about the absorption of elastic waves in the frequency range of seismological interest. That there is attenuation of elastic waves is easy to observe, as is the increase in attenuation with increasing frequency. What is difficult to measure with enough precision to support any theoretical analysis is the amount of attenuation. Bearing in mind the variable nature of geologic materials, this is not too surprising.

In any theoretical work on the amplitude of seismic signals, the effect of absorption must be taken into account. Since the experimental data are not adequate to define the appropriate parameters uniquely, the natural approach is to postulate an analytical model which satisfies the data.

The literature contains several relevant papers, those by Kolsky [1] (see also Hunter [2]) and Futterman [3] being particularly explicit, while a review article by Knopoff [4] is also very valuable. The essential point brought out in these papers is that absorption must be accompanied by dispersion [5]. In using the dispersion relations, it has proved convenient to derive an operator which can be used for convolution in the time domain. Although Kolsky derived essentially the same operator in his early paper, there does seem to be some value in repeating the calculation with appropriate comment on the applicability of the results to seismic problems.

2. ONE DIMENSIONAL PROPAGATION WITH NO ATTENUATION

In addition to absorption of energy by non-elastic processes, there is, in general, attenuation during propagation because of geometrical spreading. For simplicity we shall therefore consider one dimensional propagation so that any attenuation is entirely due to non-elastic effects. The co-ordinate system is chosen such that x represents distance and t time .

Suppose we generate at $x = 0$, $t = 0$ a pulse having Fourier components $\bar{A}(\omega)$, then the waveform $A(O,t)$ is given by

$$A(0,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{A}(\omega) \exp (i\omega t)d\omega, \qquad \ldots\ldots (1)$$

where ω is the angular frequency.

If the pulse now propagates in one dimension with constant velocity U the wave form at distance x and time t can be written

$$B(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{A}(\omega) \exp i\omega(t - x/U)d\omega. \qquad \ldots\ldots (2)$$

If we now refer time to a new origin moving with velocity V and call the new time t^1, then

$$B(x,t^1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{A}(\omega) \exp i\omega(t - x/U + x/V)d\omega. \qquad \ldots\ldots (3)$$

Comparing (1) and (3) we note that if V = U, $B(x, t^1) = A(0,t)$; which is obvious from physical considerations.

3. OBSERVATION AND THE CONSTANT Q POSTULATE

In practice it is found that absorption of energy takes place during propagation and we may write quite generally that

$$|\bar{B}(x,\omega)| = |\bar{A}(0,\omega)| \exp (- \alpha\omega x). \qquad \ldots\ldots (4)$$

Experience shows that high frequencies are preferentially absorbed and to the degree of experimental accuracy obtainable the absorption coefficient α is usually found to be independent of frequency over quite wide frequency bands.

If we postulate that as an identity α = constant, then certain complications arise. Both Futterman [3] and Kolsky [1] show that if we require a theory which is both linear (i.e., one which obeys the principle of superposition, and has the attendant mathematical advantages) and obeys the principle of causality, then:-

(a) There must be a low frequency cut-off below which the absorption coefficient is not constant but decreases.

(b) There must be a phase shift, i.e., dispersion must occur.

The first condition is relatively unimportant. Since all observations refer to some finite bandwidth we simply take the low frequency cut-off, ω_0, well below the lowest frequency of interest, so that the fact that there is this finite cut-off frequency becomes of purely academic importance.

The second condition, that there is necessarily some phase shift, is of very considerable importance. From Section 2, we had, for no absorption

$$\overline{B}(x,\omega) = \overline{A}(0,\omega) \exp - (i\omega x/U), \qquad \ldots\ldots (5)$$

where U is a constant. Now with absorption we must write

$$\overline{B}(x,\omega) = \overline{A}(0,\omega) \exp (i\omega K x), \qquad \ldots\ldots (6)$$

where $K = i\alpha - 1/C(\omega)$, in which α is, by definition, constant for $\omega > \omega_0$ and $C(\omega)$, the phase velocity at (angular) frequency ω, is a function of ω.

Although the variation of C with ω is probably too small to measure by direct means, the fact that there is a variation has fundamental implications to the subsequent analysis. Essentially it derives from the causality condition and is analogous to the similar theorem in electronic circuit theory whereby a filter cannot give an output before it receives an input. Thus, as is shown in many texts (e.g., Mason and Zimmermann [6]), the real and imaginary parts of the filter transfer function (corresponding to α and C respectively) are intimately related via the Hilbert transform.

Futterman's analysis is analytical and makes no appeal to physical processes, whereas Kolsky's analysis is based on the concept of linear elastoviscosity [2]. Their results are for practical purposes identical, although in the following analysis Futterman's notation will be used.

We first define Q_0 by the relation

$$\alpha = 1/2Q_0C_0, \qquad \ldots\ldots\ldots (7)$$

where C_0, a constant, is the velocity of the very low frequency waves ($\omega < \omega_0$) which suffer no absorption. Since α and C_0 are both constants, Q_0 is a constant also. Then the phase velocity, $C(\omega)$, of the wave having frequency ω is given by

$$C = C_0 \left[\frac{1 - \ln(\gamma\omega/\omega_0)}{\pi Q_0}\right]^{-1}, \qquad \ldots\ldots (8)$$

where γ is Euler's constant.

This expresssion is derived by Futterman for a particular model of the behaviour of α for $\omega < \omega_0$. The details need not concern us; it is sufficient to accept that C varies but slowly with ω according to the logarithmic law.

Note that for a given frequency, ω, the conventional definition of Q relates the absorption coefficient α (which is defined as being constant in the frequency range of interest) to the phase velocity C (which is a function of ω) by the equation

$$Q = 1/2\alpha C, \qquad \ldots\ldots\ldots (9)$$

so that to say that we have a constant Q model is strictly incorrect. However to insist on the difference between constant Q and constant Q_o would be pedantic and henceforth the subscript is omitted.

From equations (6), (7) and (8) we can therefore write

$$B(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{A}(0,\omega)\ \exp\ (-\ \omega x/2QC)\ \exp\ i\omega\left[t - \frac{x}{C_o}\left\{1 - \frac{\ln(\gamma\omega/\omega_o)}{\pi Q}\right\}\right]d\omega. \qquad \ldots\ldots (10)$$

If instead of going to the frequency domain as in equation (10) we use convolution in the time domain, we can write

$$B(x,t) = \int_{-\infty}^{+\infty} A(0,\tau)I(x,t-\tau)d\tau, \qquad \ldots\ldots (11)$$

where I(x,t) is derived from equation (10) by putting $A(0,\omega) = 1$, i.e., I(x,t) is the response to unit impulse, $\delta(t)$, applied at $x = 0$. Our aim now is to evaluate the impulse response.

4. THE NORMALISED IMPULSE RESPONSE

Consider the impulse response derived in the previous section

$$I(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-\ \omega x/2QC_o)\ \exp\ i\omega\left[t - \frac{x}{C_o}\left\{\frac{1 - \ln(\gamma\omega/\omega_o)}{\pi Q}\right\}\right]dw. \qquad \ldots\ldots (12)$$

Since the function is a real function of space and time $K(\omega) = K^*(-\omega)$ and we can write

$$I(x,t) = \frac{1}{\pi}\int_{0}^{+\infty} \exp(-\omega x/2QC_o)\ \cos\ \omega\left[t - \frac{x}{C_o}\left\{1 - \frac{\ln(\gamma\omega/\omega_o)}{\pi Q}\right\}\right]dw. \qquad \ldots\ldots (13)$$

As in Section 2 it would now be convenient to refer I(x,t) to a moving frame of reference. Examination of equation (13) shows that choosing a frame of reference moving with constant velocity C_o is one possibility. Kolsky [1] suggests using a frame of reference which travels with a velocity

$$V = C_o \left[\frac{1 + \ln(x/x_o)}{\pi Q}\right], \qquad \ldots\ldots \quad (14)$$

where x_o is an arbitrary constant. This has a very important result that it introduces similarity (see later) but the choice of x_o has no physical basis.

The question which then arises is, can the advantages of similarity be preserved with a more physically satisfying choice of velocity?

In any particular case where the integral in equation (13) is evaluated numerically, some limit will necessarily be set on accuracy. For instance we may choose to work to an accuracy of 1 part in 10^6. Then to this degree of approximation there will be for each case some value of frequency, ω' say, above which all contributions to the integral will be negligible, e.g., $\int_{\omega'}^{\infty}$ of equation (13) is less than 10^{-6}. Examination of equation (13) shows that it is more convenient to work in terms of the dimensionless parameter

$$\frac{\omega^1 x}{QC_o} = D, \text{ say.} \qquad \ldots\ldots \quad (15)$$

The accuracy of the solution is then directly related to D, a value of 10π being adequate for most calculations.

The velocity to be used as the velocity of the frame of reference should be related to ω'. Neither the phase velocity

$$C(\omega^1) = C_o \left[\frac{1 - \ln(\gamma\omega^1/\omega_o)}{\pi Q}\right]^{-1} \qquad \ldots\ldots \quad (16)$$

nor the group velocity

$$U(\omega^1) = C_o \left[1 - \frac{1}{\pi Q}\left\{1 + \ln(\gamma\omega^1/\omega_o)\right\}\right]^{-1} \qquad \ldots\ldots \quad (17)$$

are appropriate, the signal commencing before the origin in both cases. If however, we heuristically define an average "signal velocity" $S(\omega')$ from the equation

$$\frac{x}{S(\omega^1)} = \int_o^x \frac{dx}{U(\omega^1)}, \qquad \ldots\ldots \quad (18)$$

then it can readily be shown that

$$S(\omega^1) = C_o \left[1 - \frac{1}{\pi Q}\left\{2 + \ln(\gamma\omega^1/\omega_o)\right\}\right]^{-1}. \qquad \ldots\ldots (19)$$

It is interesting to note that if $U(\omega')$ and $C(\omega')$ are substituted in equation (18) in place of $S(\omega')$ and $U(\omega')$ respectively then the solution for $U(\omega')$ is in fact equation (17), which can equally be derived from the more familiar equation

$$U(\omega) = C - \lambda dC/d\lambda. \qquad \ldots\ldots\ldots (20)$$

This velocity, $S(\omega')$, is then the pulse velocity, and is a function of distance. Note that the pulse velocity is to some extent arbitrary because it depends on the highest frequency resolvable, but the resolution changes so rapidly with frequency while the velocity changes so slowly with frequency that no real problem arises.

Referred to an origin moving with velocity $S(\omega')$ the impulse response can be written

$$I_o(t^1) = \frac{1}{\pi}\int_o^{\omega^1} \exp\,(-\,\omega x/2QC_o)\cos\omega\left[t^1 + \frac{x}{\pi QC_o}\left\{\ln(\omega/\omega^1) - 2\right\}\right]d\omega, \qquad \ldots\ldots(21)$$

where $\omega' = (QC_oD)/x$.

This equation is a function of $\omega x/QC_o$ only and further simplification is therefore possible.

5. THE SIMILARITY SOLUTION, E(g)

The principle of similarity asserts that if $G_1(m\omega) = G_2(\omega)$ and $f(t)$ is the time transform of G_1, then the time transform of G_2 is

$$\frac{1}{m}f(t/m). \qquad \ldots\ldots\ldots (22)$$

Applying this principle to equation (18) we can write, ignoring the superscript,

$$I_o(t) = \frac{QC_o}{x}\,E(x/C_oQ),$$

where E(g)

$$= \frac{1}{\pi}\int_{o}^{D} \exp(-h/2) \cos h\left[g + \frac{1}{\pi}\left\{\ln(h/D) - 2\right\}\right]dh, \qquad \ldots\ldots (23)$$

where g is the dimensionless time

$$g = t \div (x/QC_o)$$

and h is the dimensionless (angular) frequency, $h = \omega(x/QC_o)$.

This then is the impulse response which is to be calculated.

Note that to a good approximation x/C_o can be written as T the travel time. This is particularly useful when considering cases where Q and C_o vary with position for then we can replace, $\int \frac{dx}{QC_o}$ along the propagation path by T/Q_{av} where T is the travel time and Q_{av} an effective "average" value for the whole path travelled. Thus in practice we estimate T/Q and use the function (Q/T)E(T/Q) as the convolution operator which allows for absorption.

6. EVALUATION OF THE FUNCTION E(g)

The function

$$E(g) = \frac{1}{\pi}\int_{o}^{D}\left[\exp(-h/2) \cos h(g + \left\{\frac{\ln(h/D) - 2}{\pi}\right\}\right]dh \qquad \ldots\ldots (24)$$

was evaluated by standard Fourier series techniques, replacing the integral by a summation.

Two parameters are required, the maximum and minimum (excluding zero) frequencies. The maximum, or Nyquist, frequency FNYQ cycles is selected directly and gives the maximum value of h according to the equation

$$HMAX = 2\pi FNYQ \equiv D.$$

This parameter defines the sampling interval in the g (≡time) domain, i.e.,

$$DELG = 1/(2.FNYQ).$$

The minimum frequency is defined indirectly as the reciprocal of the fundamental period (GMAX) which is in turn derived as a number, Z, times the sampling interval. Thus,

$$GMAX = Z.DELG$$

$$FMIN = 1/GMAX = 1/(Z.DELG)$$

and $HMIN = 2\pi.FMIN = 2\pi/(Z.DELG) = 4\pi.FNYQ/Z.$

The integral is now replaced by a summation using increments of h, DELH, equal to the minimum frequency.

Thus,

$$E'(R.DELG) = DELH\left[0.5 + \sum_{n=1}^{n=Z/2} \exp(-N.DELH) \cos N.DELH\left[R.DELG + \frac{1}{\pi}\left\{\ln\frac{(N.DELH)}{HMAX} - 2\right\}\right]\right], \quad \ldots\ldots (25)$$

where R takes integral values from O to Z, where the dash is used to denote a calculated value as opposed to an absolute value.

The first term inside the brackets, 0.5, takes account of the zero frequency component which is unaltered during the pulse transmission. This ensures that

$$\int_{R=o}^{Z} E'(R.DELG).\ DELG = 1,$$

i.e., the area under the $E'(g)$ curve is always unity, as required by the condition that very low frequencies are unattenuated. By using the series expansion we have also forced $E'(O) = E'(GMAX)$. In general the series expansion is an approximation to the integral, hence the use of the superscript to indicate an approximate evaluation.

7. RESULTS

The expression for $E'(g)$ was evaluated for several combinations of the parameters FNYQ and Z. The results for FNYQ = 5, Z = 1000 are given in Table 1 and plotted in Figure 1 together with the curve for FNYQ = 2, Z = 80. Note that $E'(g)$ is positive for all values of g. If, therefore, one imagines convolving $E'(g)$ with <u>any</u> input waveform, it is clear that the model cannot provide a mechanism for the inversion of first motion. Despite the fact that phase changes of π are produced, the changes are virtually linearly dependent upon frequency (from equation (21)) and a time shift rather than inversion occurs.

Inaccuracies in $E'g$ (i.e., differences from $E(g)$) arise in three ways. Firstly, there is an error due to replacing an integral over frequency from zero to infinity by an integral from zero to some finite maximum, FNYQ. Secondly, there are inaccuracies in evaluation of the integral by a summation. Finally, there is the inaccuracy due to the requirement that the integral over the duration of the transient should be unity. The first type of error, which must be less than exp ($-\pi$FNYQ), can be reduced by increasing FNYQ, the second and third by increasing the duration of the transient, i.e., Z. In general the third type of error is numerically the most significant, but appears simply as a base line shift. This effect is well demonstrated in Figure 2, where $E'(g)$ is plotted on a logarithmic scale for different values of Z.

The significance of the second source of inaccuracy can be evaluated by examining how first differences vary with Z, thus removing the larger base line shift. For FNYQ = 5, Z = 1000 the errors are of the order of 10^{-7}, which are comparable with the first type of inaccuracy for FNYQ = 5. If therefore the value of $E'(O)$ is subtracted from the $E'(g)$ values given in Table 1, the results represent the values of $E(g)$ to an accuracy of at least the last digit.

The problem of defining an arrival time is well illustrated in Figures 1 and 2. The plotting accuracy in Figure 1 is about 10^{-3}, and to this accuracy FNYQ = 2, Z = 80 is adequate. The time origin is chosen as the arrival of FNYQ travelling with the signal velocity as defined by equation (19), the corresponding group and phase arrival times being $1/\pi$ and $2/\pi$ units respectively. For FNYQ = 5 the origin is referred to the signal velocity for FNYQ = 5. In Figure 1 the plotting accuracy is not sufficient to show the early high frequency arrivals, but Figure 2 demonstrates how the apparent onset gets earlier with increasing resolution.

8. CONCLUSIONS

The main purpose of this report was to calculate the impulse response of a system satisfying the constant Q hypothesis. There seems to be no practical justification for regarding the models of Futterman and Kolsky as different, and although Futterman's analysis has been more widely quoted, Kolsky's work contains several valuable features which can be applied directly to Futterman's model. In particular the principle of similarity means that a single operator in the time domain can be scaled for use with any values of the independent variable. In a homogeneous medium this variable is the distance divided by C_oQ, but for an inhomogeneous model the variable can be replaced by the parameter $\int (C_oQ)^{-1}dx$, where the integration is taken over the whole path. Although for the model the velocities are a continuous function of frequency, the changes are so slight that for practical purposes the parameter can be written as T/Q_{av}, where T is the travel time. Nevertheless the change of velocity with frequency, though small, does present some interesting problems in rigorously defining the arrival time. The arrival time is a function of resolution, and for practical applications it corresponds to the "signal" velocity of a frequency of about (2Q/T) cycles per second. Perhaps the most important observation is that, since the operator is positive for all time, there can be no reversal of first motion.

The operator as calculated is suitable for taking account of absorption by convolution in the time domain. In many cases it is more convenient to work in the frequency domain throughout, but even then a knowledge of the impulse response of the absorptive part of the system will help assess its importance. In particular the above nomenclature may assist in avoiding that irritating feature of numerical calculation, the precursor.

APPENDIX A

Although as was stated in the text, pulse experiments cannot be expected to show the dispersion associated with absorption, there does appear to be possibility of an observable effect. Suppose the model is physically realised. The percentage difference in velocity between two periods T_1 and T_2 is approximately $30Q^{-1} \log_e T_2/T_1$. Now in current geophysical techniques applied to the earth, T varies between 1 s for body waves and 1 h for the earth's fundamental oscillations, giving the $\log_e T_2/T_1$ term value of up to 10. In the "low velocity" or "low Q" layer beneath the Moho, Q may be as low as 40 for shear waves and 100 for P waves. Therefore we might expect that, since velocity decreases with period, the low velocity layer becomes more pronounced for S waves than for P waves, and more pronounced for the longer periods. It is certainly true that over the past few years, as longer period waves have been observed, the evidence of the low velocity layer has increased substantially. Although the low velocity (or Q) layer is of relatively small thickness, the accuracy of 1 part in 10^4 or 10^5 with which fundamental oscillations are observed does raise the possibility that changes of velocity with period could be observed. In essence it means that we would need a velocity depth structure which is frequency dependent.

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TABLE 1

$E'(g)$ for FNYQ = 5, Z = 1000

g	0	0.1	0.2	0.3	0.4
E' (g)	0.00005	0.00005	0.00006	0.00016	0.00158
g	0.5	0.6	0.7	0.8	0.9
E' (g)	0.01060	0.04149	0.10813	0.20906	0.32507
g	1.0	1.1	1.2	1.3	1.4
E' (g)	0.43128	0.50986	0.55443	0.56760	0.55647
g	1.5	1.6	1.7	1.8	1.9
E' (g)	0.52902	0.49219	0.45116	0.40948	0.36930
g	2.0	2.1	2.2	2.3	2.4
E' (g)	0.33186	0.29769	0.26695	0.23954	0.21524
g	2.5	2.6	2.7	2.8	2.9
E' (g)	0.19376	0.17481	0.15810	0.14334	0.13031
g	3.0	3.1	3.2	3.3	3.4
E' (g)	0.11876	0.10853	0.09943	0.09132	0.08408
g	3.5	3.6	3.7	3.8	3.9
E' (g)	0.07759	0.07177	0.06653	0.06180	0.05753
g	4.0	4.2	4.4	4.6	4.8
E' (g)	0.05366	0.04693	0.04153	0.03664	0.03266
g	5.0	5.2	5.4	5.6	5.8
E' (g)	0.02927	0.02636	0.02385	0.02167	0.01977
g	6.0	6.2	6.4	6.6	6.8
E' (g)	0.01810	0.01663	0.01532	0.01416	0.01312
g	7.0	7.2	7.4	7.6	7.8
E' (g)	0.01219	0.01136	0.01060	0.00992	0.00930
g	8.0	8.5	9.0	9.5	10.0
E' (g)	0.00873	0.00752	0.00654	0.00574	0.00508
g	10.5	11.0	11.5	12.0	12.5
E' (g)	0.00453	0.00406	0.00365	0.00331	0.00301
g	13.0	13.5	14.0	14.5	15.0
E' (g)	0.00275	0.00253	0.00233	0.00215	0.00199
g	16.0	17.0	18.0	19.0	20.0
E' (g)	0.00173	0.00151	0.00133	0.00119	0.00106
g	22.0	24.0	26.0	28.0	30.0
E' (g)	0.00087	0.00072	0.00061	0.00053	0.00046
g	32.0	34.0	36.0	38.0	40.0
E' (g)	0.00040	0.00036	0.00032	0.00029	0.00026
g	50.0	60.0	70.0	80.0	90.0
E' (g)	0.00017	0.00012	0.00010	0.00008	0.00006

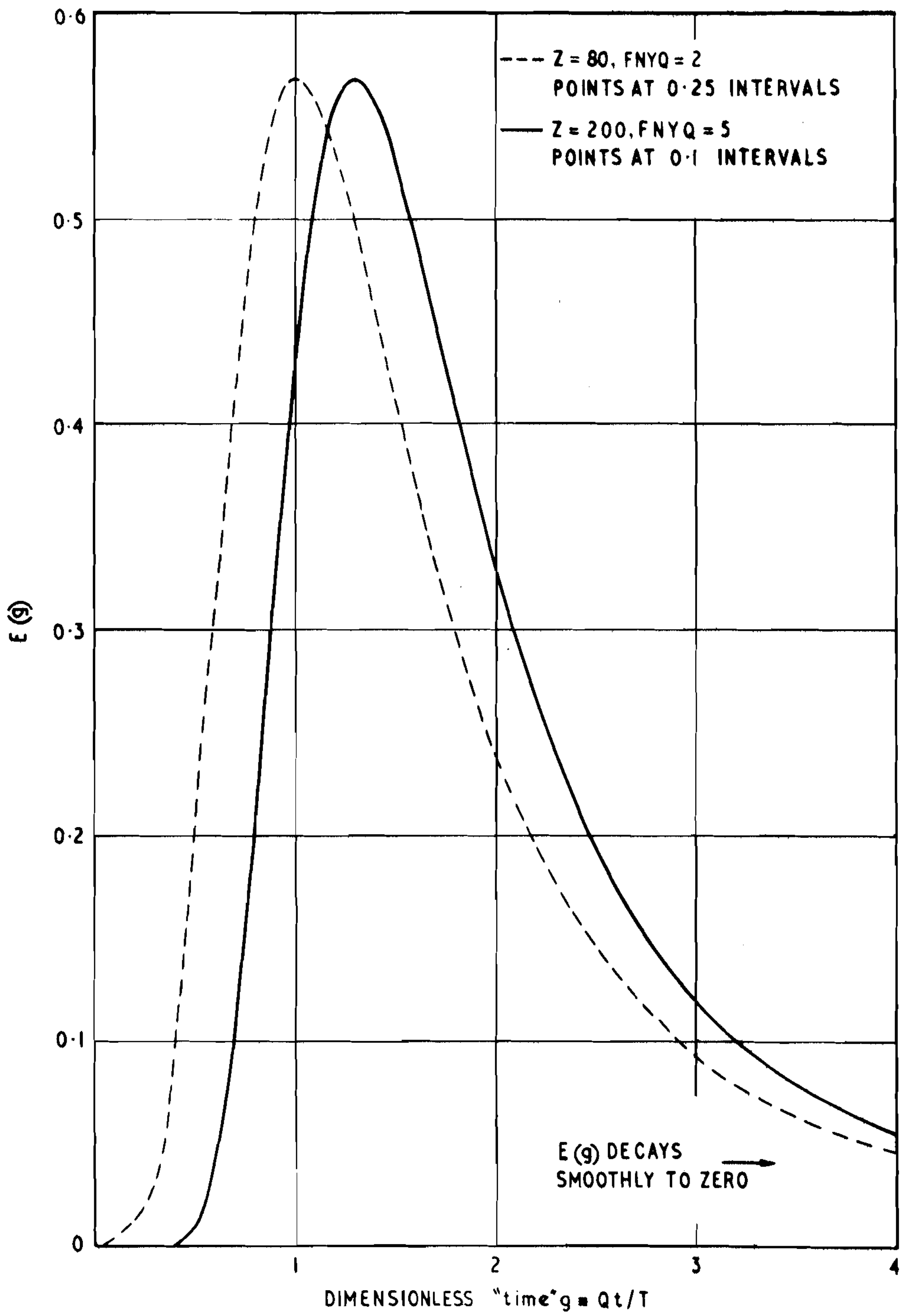

FIGURE 1. THE FUNCTION E(g) TO g = 4

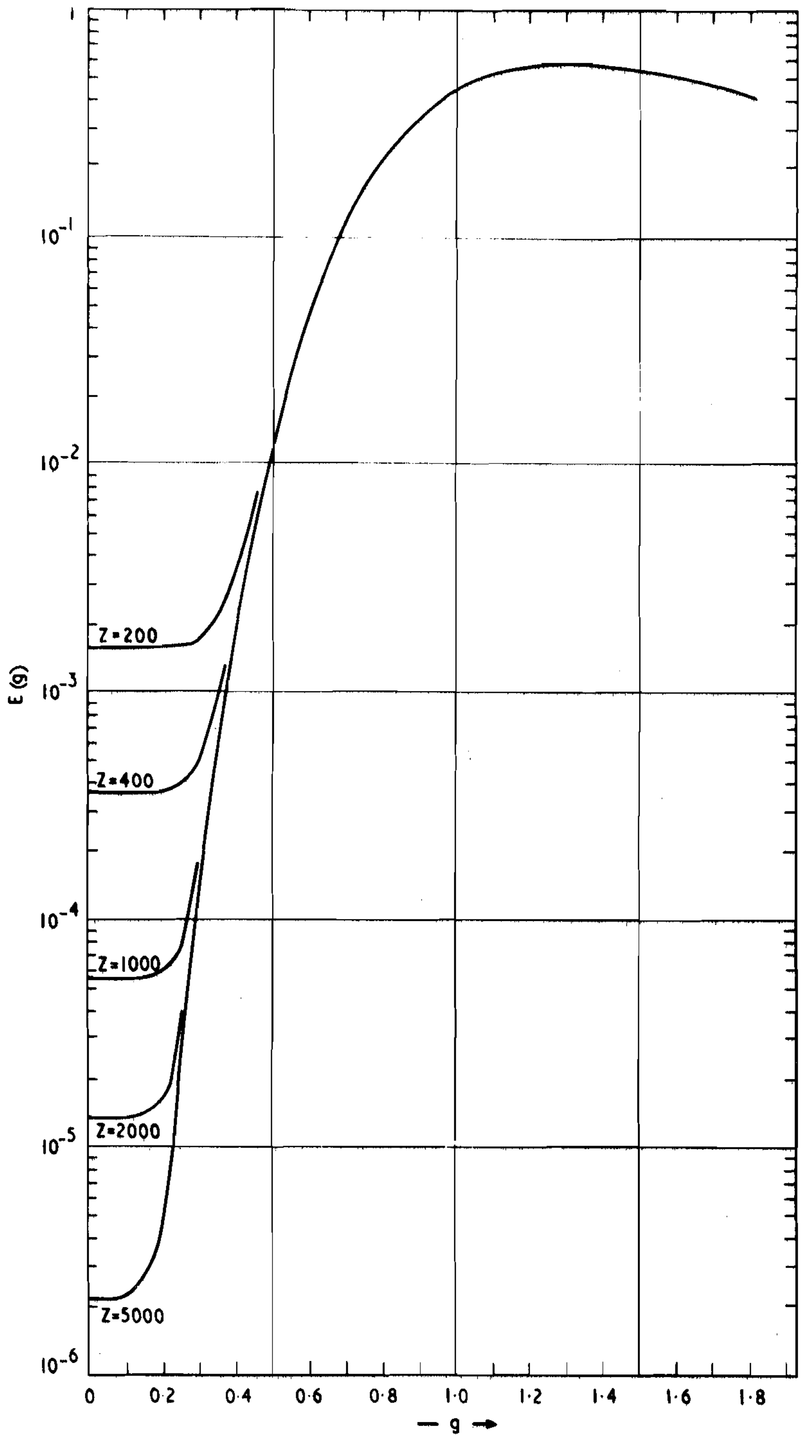

FIGURE 2. LOGARITHMIC PLOT OF E(g) SHOWING EFFECT OF INCREASING Z